

Convergence of Kähler-Ricci flow on lower dimensional algebraic manifolds of general type

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Abstract

In this paper, we prove that the L^4 -norm of Ricci curvature is uniformly bounded along a Kähler-Ricci flow on any minimal algebraic manifold. As an application, we show that on any minimal algebraic manifold M of general type and with dimension $n \leq 3$, any solution of the normalized Kähler-Ricci flow converges to the unique singular Kähler-Einstein metric on the canonical model of M in the Cheeger-Gromov topology.

1 Introduction

The purpose of this note is to prove the following

Theorem 1.1. *Let M be a smooth minimal model of general type with dimension $n \leq 3$ and $\omega(t)$ be a solution to the normalized Kähler-Ricci flow*

$$\frac{\partial}{\partial t}\omega = -\text{Ric} - \omega. \quad (1.1)$$

Then $(M, \omega(t))$ converges in the Cheeger-Gromov sense to the unique singular Kähler-Einstein metric on the canonical model of M .

Here, by a smooth minimal model, we mean an algebraic manifold M with nef canonical bundle K_M . The theorem should remain true in higher dimensional case; cf. Conjecture 4.1 in [27]. Assuming the uniform bound of Ricci curvature, the conjecture is confirmed by Guo [18]. On the other hand, it has been known since Tsuji [32] and Tian-Zhang [31] the convergence of the Kähler-Ricci flow in the current sense and the smooth convergence on the ample locus of the canonical class.

Applying the L^2 bound of Riemannian curvature (cf. Section 3) and Kähler-Einstein condition, we can say more about the limit singular space M_∞ . When $n = 2$, by a

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classical argument of removing isolated singularities following Anderson [1], Bando-Kasue-Nakajima [4] and Tian [29], we can show that the limit is a smooth Kähler-Einstein orbifold with finite orbifold points. When $n = 3$, by the argument of Cheeger-Colding-Tian [11] or Cheeger [7], we have that the 2-dimensional Hausdorff measure of the singular set is finite. Moreover, it follows from the parabolic version of the partial C^0 -estimate (see [30]) that the limit M_∞ is a normal variety and there is a natural holomorphic map from M onto M_∞ . This actually implies that M_∞ is the canonical model of M .

The proof of our theorem relies on a uniform L^4 bound of Ricci curvature under the Kähler-Ricci flow on a smooth minimal model together with the diameter boundedness of the limit singular Kähler-Einstein space (in the case of general type) proved by Song [26]. From these we derive a uniform local noncollapsing estimate of Kähler-Ricci flow on a minimal model of general type and the Gromov-Hausdorff convergence follows.

In Section 2, we present a short discussion on manifolds with integral bounded Ricci curvature, with emphasis on a uniform local noncollapsing estimate which is essential in extending the regularity theory of Cheeger-Colding and Cheeger-Colding-Tian. Then, in Section 3, we give a proof of our theorem by establishing a uniform L^4 Ricci curvature estimate under the Kähler-Ricci flow.

After we completed the first draft of this paper, we learned that Guo-Song-Weinkove obtained a different proof for the 2-dimensional case of Theorem 1.1 (see [19]).

2 Manifolds with L^p bounded Ricci curvature

We recall the relative volume comparison formula relative to the L^p Ricci curvature, p bigger than half dimension, which is due to Petersen-Wei [21]. It implies a uniform local noncollapsing property for the Kähler-Ricci flow on lower dimensional minimal models of general type. We will use this to prove the convergence of Kähler-Ricci flow on such manifolds.

Let (M, g) be a complete Riemannian manifold of (real) dimension m . For any $\kappa \in \mathbb{R}$ denote by B_r^κ a metric ball of radius r in the space form of dimension m with sectional curvature κ and by $\text{vol}(B_r^\kappa)$ its volume. Then we have

Theorem 2.1 ([21]). *For any $p > \frac{m}{2}$ and $\Lambda < 0$, there exists $C = C(m, p, \Lambda)$ such that the followings hold. First of all, for any $x \in M$ and $r > 0$,*

$$\frac{d}{dr} \left(\frac{\text{vol}(B_r(x))}{\text{vol}(B_r^\Lambda)} \right)^{\frac{1}{2p}} \leq \frac{Cr^{2p}}{\text{vol}(B_r^\Lambda)} \left(\int_{B_r(x)} |(\text{Ric} - (m-1)\Lambda g)_-|^p dv \right)^{\frac{1}{2p}}, \quad (2.1)$$

where we define

$$(\text{Ric} - (m-1)\Lambda g)_- = \max_{|v|=1} (0, -\text{Ric}(v, v) + (m-1)\Lambda)$$

pointwisely. In particular, for any $r_2 > r_1 > 0$,

$$\left(\frac{\text{vol}(B_{r_2}(x))}{\text{vol}(B_{r_2}^\Lambda)} \right)^{\frac{1}{2p}} - \left(\frac{\text{vol}(B_{r_1}(x))}{\text{vol}(B_{r_1}^\Lambda)} \right)^{\frac{1}{2p}} \leq C \left(\int_{B_{r_2}(x)} |(\text{Ric} - (m-1)\Lambda g)_-|^p dv \right)^{\frac{1}{2p}}. \quad (2.2)$$

Then, by letting $r_1 \rightarrow 0$ it gives, for any $r > 0$,

$$\frac{\text{vol}(B_r(x))}{\text{vol}(B_r^\Lambda)} \leq 1 + C \int_{B_r(x)} |(\text{Ric} - (m-1)\Lambda g)_-|^p dv. \quad (2.3)$$

The following corollary gives a kind of uniform local noncollapsing property on manifolds with integral Ricci curvature bound; see [22] and [16] for similar volume doubling estimates.

Corollary 2.2. *For any $\Lambda < 0$ and $p > \frac{m}{2}$, there exists $\varepsilon = \varepsilon(m, p, \Lambda) > 0$ such that the following holds. Fix a base point $x_0 \in M$. For any $x \in M$ with $\text{dist}(x_0, x) = d$, if*

$$\frac{1}{\text{vol}(B_1(x_0))} \int_{B_{2d+1}(x_0)} |(\text{Ric} - (m-1)\Lambda g)_-|^p dv \leq \frac{\varepsilon}{\text{vol}(B_{d+1}^\Lambda)}, \quad (2.4)$$

then,

$$\frac{\text{vol}(B_r(x))}{r^m} \geq \frac{\text{vol}(B_1(x_0))}{2 \text{vol}(B_{d+1}^\Lambda)}, \quad \forall r \leq 1. \quad (2.5)$$

Proof. By (2.2), for any $r \leq 1$,

$$\begin{aligned} \left(\frac{\text{vol}(B_r(x))}{\text{vol}(B_r^\Lambda)} \right)^{\frac{1}{2p}} &\geq \left(\frac{\text{vol}(B_{d+1}(x))}{\text{vol}(B_{d+1}^\Lambda)} \right)^{\frac{1}{2p}} - C \left(\int_{B_{d+1}(x)} |(\text{Ric} - (m-1)\Lambda g)_-|^p dv \right)^{\frac{1}{2p}} \\ &\geq \left(\frac{\text{vol}(B_1(x_0))}{\text{vol}(B_{d+1}^\Lambda)} \right)^{\frac{1}{2p}} - C \left(\int_{B_{2d+1}(x_0)} |(\text{Ric} - (m-1)\Lambda g)_-|^p dv \right)^{\frac{1}{2p}}. \end{aligned}$$

where $C = C(m, p, \Lambda)$. If (2.4) holds for some $\varepsilon = \varepsilon(m, p, \Lambda)$ sufficiently small, then

$$\left(\frac{\text{vol}(B_r(x))}{\text{vol}(B_r^\Lambda)} \right)^{\frac{1}{2p}} \geq \frac{1}{2^{2p}} \left(\frac{\text{vol}(B_1(x_0))}{\text{vol}(B_{d+1}^\Lambda)} \right)^{\frac{1}{2p}},$$

which is exactly the estimate (2.5). \square

The lemma suggests a condition for Gromov-Hausdorff convergence. Suppose we have a sequence of complete Riemannian manifolds (M_i, g_i) of dimension m such that

$$\int_M |(\text{Ric}_{g_i} - (m-1)\Lambda g_i)_-|^p dv_{g_i} \rightarrow 0 \quad (2.6)$$

for some $p > \frac{m}{2}$, $\Lambda > 0$, and

$$\text{vol}_{g_i}(B_1(x_i)) \geq v \quad (2.7)$$

uniformly for some $v > 0$ and $x_i \in M_i$, then along a subsequence, the manifolds (M_i, g_i) are uniformly locally noncollapsing on $B_r(x_i)$ for any fixed $r > 0$. Thus, we can apply Gromov precompactness theorem to get a noncollapsing limit of (M_i, g_i, x_i) in the Gromov-Hausdorff topology. Furthermore, as showed in [30], one can extend the regularity theory

of Colding [12], Cheeger-Colding [8, 9], Cheeger-Colding-Tian [11] and Colding-Naber [13] in this setting. If in addition we replace (2.6) by

$$\int_M |\text{Ric}_{g_i} - (m-1)\Lambda g_i|^p dv_{g_i} \rightarrow 0, \quad (2.8)$$

then Anderson's harmonic radius estimate [2] can also be applied. In summation, we can follow the arguments in [22] and [30] to prove

Theorem 2.3. *Let (M_i, g_i) be a sequence of Riemannian manifolds satisfying (2.7) and (2.8) for some $p > \frac{m}{2}$ and $\Lambda, v > 0$. Then, passing to a subsequence if necessary, (M_i, g_i, x_i) converges in the Cheeger-Gromov sense to a limit length space $(M_\infty, d_\infty, x_\infty)$ such that*

- (1) *for any $r > 0$ and $y_i \in M_i$ with $y_i \rightarrow y_\infty \in M_\infty$ we have*

$$\text{vol}(B_r(y_i)) \rightarrow \mathcal{H}^m(B_r(y_\infty)), \quad (2.9)$$

where \mathcal{H}^m denotes the m -dimensional Hausdorff measure;

- (2) *$M_\infty = \mathcal{R} \cup \mathcal{S}$ such that \mathcal{S} is a closed set of codimension ≥ 2 and \mathcal{R} is convex in M_∞ ; \mathcal{R} consists of points whose tangent cones are \mathbb{R}^m ;*
- (3) *the convergence on \mathcal{R} takes place in the C^α topology for any $0 < \alpha < 2 - \frac{m}{p}$;*
- (4) *the tangent of any $y \in M_\infty$ is a metric cone which splits off lines isometrically; the tangent cone has the same properties presented in (2) and (3);*
- (5) *the singular set of \mathcal{S} has codimension ≥ 4 if each M_i is Kählerian.*

3 Kähler-Ricci flow on minimal models

3.1 L^4 bound of Ricci curvature under Kähler-Ricci flow on minimal models

Let M be a smooth minimal model. If the Kodaira dimension equals 0, the manifold is Calabi-Yau and any Kähler-Ricci flow on M converges smoothly to a Ricci flat metric [6]. We assume from now on that the Kodaira dimension of M is positive. Then we consider the normalized Kähler-Ricci flow

$$\frac{\partial}{\partial t} \omega(t) = -\text{Ric}_{\omega(t)} - \omega(t), \quad \omega(0) = \omega_0. \quad (3.1)$$

It is well-known that the solution exists for all time $t \in [0, \infty)$ [31].

We shall prove the following theorem.

Theorem 3.1. *Suppose M has positive Kodaira dimension and K_M is semi-ample. Then there is a constant C depending on ω_0 such that*

$$\int_t^{t+1} \int_M |\text{Ric}_{\omega(s)}|^4 \omega(s)^n ds \leq C, \quad \forall t \in [0, \infty). \quad (3.2)$$

Moreover, for any $0 < p < 4$ we have

$$\int_t^{t+1} \int_M |\text{Ric}_{\omega(s)} + \omega(s)|^p \omega(s)^n ds \rightarrow 0, \text{ as } t \rightarrow \infty. \quad (3.3)$$

We start with some general set-up following [28]. Since K_M is semi-ample, a basis of $H^0(M; K_M^\ell)$ for some large ℓ gives rise to a holomorphic map

$$\pi : M \rightarrow \mathbb{C}P^N, \quad N = \dim H^0(M; K_M^\ell) - 1. \quad (3.4)$$

Let ω_{FS} be the Fubini-Study metric on $\mathbb{C}P^N$ and put

$$\chi = \frac{1}{\ell} \pi^* \omega_{FS} \in [K_M]. \quad (3.5)$$

Choose a smooth volume form Ω such that $\text{Ric}(\Omega) = -\chi$. Put

$$\hat{\omega}_t = e^{-t} \omega_0 + (1 - e^{-t}) \chi.$$

It represents a Kähler metric in the class $[\omega(t)]$ and write

$$\omega(t) = \hat{\omega}_t + \sqrt{-1} \partial \bar{\partial} \varphi(t) \quad (3.6)$$

for a family of smooth functions $\varphi(t)$. Then the Kähler-Ricci flow (3.1) is equivalent to

$$\frac{\partial \varphi}{\partial t} = \log \frac{e^{(n-\kappa)t} (\hat{\omega}_t + \sqrt{-1} \partial \bar{\partial} \varphi)^n}{\Omega} - \varphi. \quad (3.7)$$

Lemma 3.2 ([28]). *There exists $C_i = C_i(\omega_0, \chi)$, $i = 1, 2$, such that*

$$\|\varphi(t)\|_{C^0} + \|\varphi'(t)\|_{C^0} \leq C_1, \quad \forall t \geq 0 \quad (3.8)$$

and

$$\chi \leq C_2 \omega(t), \quad \forall t \geq 0. \quad (3.9)$$

Let $u = \varphi + \varphi'$ for any time t .

Lemma 3.3 ([28]). *There exists $C_3 = C_3(\omega_0, \chi)$ such that*

$$\|\nabla u(t)\|_{C^0} + \|\Delta u(t)\|_{C^0} \leq C_3, \quad \forall t \geq 0. \quad (3.10)$$

When the manifold is of general type, these estimates are proved in [35]. We also observe that, from (3.7),

$$\text{Ric}_{\omega(t)} + \sqrt{-1} \partial \bar{\partial} u(t) = -\chi. \quad (3.11)$$

So, by the uniform bound of χ in terms of $\omega(t)$, to prove the L^4 bound of Ricci curvature, it suffices to show

$$\int_t^{t+1} \int_M |\partial \bar{\partial} u(s)|^4 \omega(s)^n ds \leq C_4, \quad \forall t \geq 0 \quad (3.12)$$

for some C_4 independent of t . We follow the line in [30] to prove this estimate.

As in [30] we let ∇ and $\bar{\nabla}$ denote the (1,0) and (0,1) part of the Levi-Civita connection of $\omega(t)$. Then, by the calculations in [30], we have the following lemmas

Lemma 3.4. *There exists $C_5 = C_5(\omega_0, \chi)$ such that*

$$\int_M (|\nabla \nabla u|^2 + |\nabla \bar{\nabla} u|^2 + |\text{Ric}|^2 + |Rm|^2) \omega^n \leq C_5, \forall t \geq 0. \quad (3.13)$$

Lemma 3.5 ([30]). *There exists a universal constant $C_6 = C_6(n)$ such that*

$$\int_M |\nabla \bar{\nabla} u|^4 \leq C_6 \int_M |\nabla u|^2 (|\bar{\nabla} \nabla \nabla u|^2 + |\nabla \nabla \bar{\nabla} u|^2); \quad (3.14)$$

$$\int_M (|\bar{\nabla} \nabla \nabla u|^2 + |\nabla \nabla \bar{\nabla} u|^2) \leq C_6 \int_M (|\nabla \Delta u|^2 + |\nabla u|^2 \cdot |Rm|^2). \quad (3.15)$$

The estimates in the last lemma are general facts which remain hold for any smooth function.

The following theorem gives the required estimate (3.2).

Theorem 3.6. *There exists $C_7 = C_7(\omega_0, \chi)$ such that*

$$\int_t^{t+1} \int_M (|\nabla \bar{\nabla} u|^4 + |\nabla \nabla \bar{\nabla} u|^2 + |\bar{\nabla} \nabla \nabla u|^2) \leq C_7, \forall t \geq 0. \quad (3.16)$$

Proof. By the previous two lemmas we are sufficient to prove a uniform L^2 bound of $\nabla \Delta u$. Recall the evolution of Δu , cf. (3.22) in [28],

$$\frac{\partial}{\partial t} \Delta u = \Delta \Delta u + \Delta u + \langle \text{Ric}, \partial \bar{\partial} u \rangle_\omega + \Delta (\text{tr}_\omega \chi).$$

Thus,

$$\frac{\partial}{\partial t} (\Delta u)^2 = \Delta (\Delta u)^2 - 2|\nabla \Delta u|^2 + 2(\Delta u)^2 + 2\Delta u \langle \text{Ric}, \partial \bar{\partial} u \rangle + 2\Delta u \Delta (\text{tr}_\omega \chi). \quad (3.17)$$

Integrating over the manifolds gives

$$\begin{aligned} \int_M |\nabla \Delta u|^2 &\leq \int_M \left((\Delta u)^2 + |\Delta u| |\text{Ric}| |\nabla \bar{\nabla} u| - \nabla_i \Delta u \nabla_{\bar{i}} (\text{tr}_\omega \chi) - \frac{1}{2} \frac{\partial}{\partial t} (\Delta u)^2 \right) \\ &\leq \int_M \left(\frac{1}{2} |\nabla \Delta u|^2 + (\Delta u)^2 + (\Delta u)^2 |\text{Ric}|^2 + |\nabla \bar{\nabla} u|^2 + 2|\nabla (\text{tr}_\omega \chi)|^2 \right) \\ &\quad - \frac{1}{2} \int_M (\Delta u)^2 (s + n) - \frac{1}{2} \frac{d}{dt} \int_M (\Delta u)^2. \end{aligned}$$

Applying the uniform bound of Δu and above lemma, we get

$$\int_M |\nabla \Delta u|^2 \leq C \int_M (1 + |\nabla (\text{tr}_\omega \chi)|^2) - \frac{d}{dt} \int_M (\Delta u)^2.$$

Integrating over the time interval we have

$$\int_t^{t+1} \int_M |\nabla \Delta u|^2 \leq C \int_t^{t+1} \int_M (1 + |\nabla (\text{tr}_\omega \chi)|^2) + C, \forall t \geq 0. \quad (3.18)$$

The term $|\nabla(\text{tr}_\omega \chi)|^2$ can be estimated through the Schwarz lemma. Recall the following formula

$$\Delta(\text{tr}_\omega \chi) \geq -|\text{Ric}| \text{tr}_\omega \chi - C(\text{tr}_\omega \chi)^2 + \frac{|\nabla(\text{tr}_\omega \chi)|^2}{\text{tr}_\omega \chi} \quad (3.19)$$

where C is a universal constant given by the upper bound of the bisectional curvature of ω_{FS} on $\mathbb{C}P^n$. Because $0 \leq \text{tr}_\omega \chi \leq C$ under the flow, we have

$$|\nabla(\text{tr}_\omega \chi)|^2 \leq C \left(\Delta(\text{tr}_\omega \chi) + C|\text{Ric}| + C \right).$$

Thus,

$$\int_M |\nabla(\text{tr}_\omega \chi)|^2 \leq C \int_M (1 + |\text{Ric}|) \leq C$$

uniformly. Substituting into (3.18) we get the desired estimate. \square

To prove (3.3) we use the L^2 estimate to traceless Ricci curvature following the calculation by Y. Zhang [34].

Lemma 3.7. *Under the Kähler-Ricci flow,*

$$\int_t^{t+1} \int_M |\text{Ric}_{\omega(s)} + \omega(s)|^2 \omega(s)^n ds \rightarrow 0, \text{ as } t \rightarrow \infty. \quad (3.20)$$

Proof. Recall the evolution of scalar curvature $R = \text{tr}_\omega \text{Ric}$, cf. [34],

$$\frac{\partial}{\partial t} R = \Delta R + |\text{Ric}|^2 + R = \Delta R + |\text{Ric} + \omega|^2 - (R + n).$$

The maximum principle shows that $\check{R} = \inf R$ satisfies $\frac{d}{dt} \check{R} \geq -(\check{R} + n)$, which implies immediately

$$\check{R}(t) + n \geq e^{-t} \min(\check{R}(0) + n, 0) \geq -Ce^{-t} \quad (3.21)$$

for some $C = C(\omega_0) > 0$. Then,

$$\begin{aligned} \int |\text{Ric} + \omega|^2 \omega^n &= \int \left(\frac{\partial}{\partial t} R + R + n \right) \omega^n \\ &= \frac{d}{dt} \int R \omega^n + \int (R + n)(R + 1) \omega^n \\ &= \frac{d}{dt} \int R \omega^n + \int (R + n + Ce^{-t})(R + 1) \omega^n - Ce^{-t} \int (R + 1) \omega^n \\ &\leq \frac{d}{dt} \int R \omega^n + C \int (R + n) \omega^n + Ce^{-t} \end{aligned}$$

where we used the uniform bound of scalar curvature and volume of $\omega(t)$. The integration of $R + n$ can be computed as

$$\int (R + n) \omega^n = n \int (\text{Ric} + \omega) \wedge \omega^{n-1} = n \int (-\chi + \hat{\omega}) \wedge \hat{\omega}^{n-1} = ne^{-t} \int (\omega_0 - \chi) \wedge \hat{\omega}^{n-1}.$$

Thus, $\int (R + n)\omega^n \leq Ce^{-t}$. Then we have

$$\int_0^\infty \int |\operatorname{Ric} + \omega|^2 \omega(t)^n dt \leq \lim_{t \rightarrow \infty} \int R(t) \omega(t)^n - \int R(0) \omega_0^n + C \leq C.$$

The lemma is proved by this estimate. \square

The estimate (3.3) when $2 \leq p < 4$ then is a direct consequence of the Hölder inequality

$$\int_t^{t+1} \int_M |\operatorname{Ric} + \omega|^p \leq \left(\int_t^{t+1} \int_M |\operatorname{Ric} + \omega|^4 \right)^{\frac{p-2}{2}} \left(\int_t^{t+1} \int_M |\operatorname{Ric} + \omega|^2 \right)^{\frac{4-p}{2}}.$$

When $0 < p < 2$ the estimate (3.3) is obvious.

Remark 3.8. *M. Simon presented in [25] an L^4 Ricci curvature estimate under Ricci flow on a four-manifold. Combined with the Kähler condition, his arguments can be adapted to give an alternative proof of our estimate. Another related integral bound of curvature can be found in [3].*

3.2 Cheeger-Gromov convergence

When the manifold M is of general type, the Kähler-Ricci flow (3.1) should converge in the Gromov-Hausdorff topology to a singular Kähler-Einstein metric on the canonical model; cf. Conjecture 4.1 in [27]. In this subsection we confirm this convergence without any curvature assumption in the case of the dimension less than or equal to 3.

Recall the holomorphic map $\pi : M \rightarrow \mathbb{C}P^N$ by a basis of $H^0(M; K_M^\ell)$ for some large ℓ . Its image $M_{\text{can}} = \pi(M)$ is called the canonical model of M . Let $E \subset M$ be the exceptional locus of π . Then we have

Theorem 3.9 ([31, 26]). *Let M be a smooth minimal model of general type and $\omega(t)$ be any solution to the Kähler-Ricci flow (3.1). Then,*

- (1) $\omega(t)$ converges in the current sense to a Kähler-Einstein metric ω_∞ , the convergence takes place smoothly outside the exceptional locus E ;
- (2) the metric completion of $(M \setminus E, \omega_\infty)$ is homeomorphic to M_{can} , so it is compact.

Remark 3.10. *It is known that the exceptional locus E coincides with the non-ample or non-Kähler locus of the canonical class; cf. [31] and [14].*

Suppose $\dim_{\mathbb{C}} M \leq 3$. Let $t_i \rightarrow \infty$ be a sequence of times such that

$$\int_M |\operatorname{Ric}_{\omega(t_i)} + \omega(t_i)|^{\frac{7}{2}} \omega(t_i)^n \rightarrow 0, \text{ as } i \rightarrow \infty. \quad (3.22)$$

Choose a regular point $x_0 \in M \setminus E$. The volume of the unit ball $\operatorname{vol}(B_{1, \omega(t_i)}(x_0))$ has a uniform lower bound. By the L^p extension of Cheeger-Colding-Tian theory, Theorem 2.3, we may assume that $(M, \omega(t_i), x_0)$ converges in the Cheeger-Gromov sense to a limit metric space $(M_\infty, d_\infty, x_\infty)$. Since the metric $\omega(t)$ converges smoothly on $M \setminus E$, we may view $(M \setminus E, \omega_\infty)$ as a subset of (M_∞, d_∞) through a locally isometric embedding.

Let $M_\infty = \mathcal{R} \cup \mathcal{S}$ the regular/singular decomposition of M_∞ .

Lemma 3.11. *Suppose $x \in \mathcal{R}$, $0 < \alpha < 2 - \frac{4n}{7}$. There exists $r = r(x) > 0$ such that any $x_i \in M$ converging to x has a holomorphic coordinate (z^1, \dots, z^n) on $B_{r, \omega(t_i)}(x_i)$ which satisfies*

$$\frac{1}{2} \leq (g_{k\bar{\ell}}) \leq 2, \quad \|g_{k\bar{\ell}}\|_{C^\alpha} \leq 2$$

where $g_{k\bar{\ell}} = \omega(t_i)(\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \bar{z}^\ell})$.

Proof. The existence of holomorphic coordinates is well-known. It can be constructed by a slight modification of the local harmonic coordinates. We include a short proof for the reader's convenience. First of all, by the C^α convergence on \mathcal{R} , there is a sequence of harmonic coordinate $v_i = (v_i^1, \dots, v_i^{2n})$ on $B_{r, \omega(t_i)}(x_i)$ for some $r > 0$ independent of i such that

$$\frac{3}{4} \leq (h_{pq}) \leq \frac{4}{3} \text{ and } r^{-\alpha} \|h_{pq}\|_{C^\alpha} \leq \frac{4}{3},$$

where $h_{pq} = g_i(\frac{\partial}{\partial v_i^p}, \frac{\partial}{\partial v_i^q})$, g_i is the Kähler metric of $\omega(t_i)$, and [22]

$$\int_{B_{r, \omega(t_i)}(x_i)} |\langle \frac{\partial}{\partial v^p}, \frac{\partial}{\partial v^q} \rangle - \delta_{pq}| \omega(t_i)^n \rightarrow 0, \text{ as } i \rightarrow \infty.$$

Moreover, since each $\omega(t_i)$ is Kähler, we may assume that

$$\int_{B_{r, \omega(t_i)}(x_i)} |\langle J \frac{\partial}{\partial v^p}, \frac{\partial}{\partial v^{n+q}} \rangle - \delta_{pq}| \omega(t_i)^n \rightarrow 0, \text{ as } i \rightarrow \infty,$$

for any $1 \leq p, q \leq n$, where J is the complex structure of M ; see [11, Section 9] for a discussion. Then we choose a pseudoconvex domain $B_{\frac{1}{2}r, \omega(t_i)}(x_i) \subset \Omega_i \subset B_{r, \omega(t_i)}(x_i)$ and solve the $\bar{\partial}$ equation in Ω_i :

$$\bar{\partial} f_i^p = \bar{\partial}(v_i^p + \sqrt{-1}v^{n+p}), \quad p = 1, \dots, n.$$

The domain can be chosen as the Euclidean ball in the local coordinate. The equation has solution satisfying the L^2 estimate [24, Theorem 4.3.4]

$$\int_{B_{\frac{1}{2}r, \omega(t_i)}(x_i)} |f_i^p|^2 \omega(t_i)^n \leq Cr^2 \int_{B_{r, \omega(t_i)}(x_i)} |\bar{\partial}(v_i^p + \sqrt{-1}v^{n+p})|^2 \omega(t_i)^n$$

for a universal constant C . This implies $\int_{B_{\frac{1}{2}r, \omega(t_i)}(x_i)} |f_i^p|^2 \omega(t_i)^n \rightarrow 0$ for all $1 \leq p \leq n$.

Then applying the elliptic regularity to $\Delta_{\omega(t_i)} f_i^p = 0$ we get

$$\sup_{B_{\frac{1}{4}r, \omega(t_i)}(x_i)} (|\partial f_i^p| + |\bar{\partial} f_i^p|) \rightarrow 0, \quad \forall 1 \leq p \leq n.$$

In particular, the function $w_i^p = v^p + \sqrt{-1}v^{n+p} - f_i^p$, $1 \leq p \leq n$, gives rise to the desired holomorphic coordinate on $B_{\frac{1}{4}r, \omega(t_i)}(x_i)$ whenever i is large enough. \square

Lemma 3.12. *$M_\infty \setminus (M \setminus E)$ is a closed subset of M_∞ with Hausdoeff codimension ≥ 2 .*

Proof. Notice that $M \setminus E \subset \mathcal{R}$, $M_\infty \setminus (M \setminus E) = \mathcal{S} \cup (\mathcal{R} \setminus (M \setminus E))$ where $\text{Codim } \mathcal{S} \geq 4$ by Theorem 2.3 (5). Therefore, it suffices to show that $\text{Codim } (\mathcal{R} \setminus (M \setminus E)) \geq 2$.

For any $x \in \mathcal{R} \setminus (M \setminus E)$ there exists a sequence of points $x_i \in E$ which converges to x . By above lemma, there exists local holomorphic coordinate in $B_{r, \omega(t_i)}(x_i)$ for some $r = r(x) > 0$ independent of i with required C^α estimate. The intersection $E \cap B_{r, \omega(t_i)}(x_i)$ is a local subvariety with finite volume, so passing to a subsequence, $E \cap B_{r, \omega(t_i)}(x_i)$ converges to a limit analytical set $E_\infty \subset B_{r, \omega_\infty}(x)$. Thus, $\mathcal{R} \setminus (M \setminus E)$ is an analytical set, $\text{Codim } (\mathcal{R} \setminus (M \setminus E)) \geq 2$ as desired. \square

Lemma 3.13. *(M_∞, d_∞) is isometric to the metric completion of $(M \setminus E, \omega_\infty)$.*

Proof. The lemma follows from the fact that $\text{Codim } (\mathcal{R} \setminus (M \setminus E)) \geq 2$ and the local isoperimetric constant estimate; see [10] or [23] for details. For the estimate of local isoperimetric constant, one can apply the same argument as Croke [15] by using the volume comparison of geodesic balls by Petersen-Wei (cf. [30, Corollary 2.4] or (2.3)). \square

By the compactness of the limit space by Song [26], see Theorem 3.9 (2) above, the diameters of the sequence $(M, \omega(t_i))$ are uniformly bounded.

Lemma 3.14. *The Kähler-Ricci flow $\omega(t)$ is uniformly noncollapsing in the sense that: there exists $\kappa = \kappa(n, \omega_0) > 0$ such that*

$$\text{vol}_{\omega(t)}(B_{r, \omega(t)}(x)) \geq \kappa r^{2n}, \forall x \in M, r \leq 1. \quad (3.23)$$

Proof. The lemma follows from Perelman's noncollapsing estimate to Ricci flow [20] together with the scalar curvature estimate by Z. Zhang [35]. Suppose that

$$r_i^{-2n} \text{vol}_{\omega(t_i)}(B_{r_i, \omega(t_i)}(x_i)) \rightarrow 0 \quad (3.24)$$

for a sequence of times $t_i \rightarrow \infty$ and $x_i \in M$, $r_i \leq 1$. Choose $t'_i \in [t_i - 2, t_i - 1]$ such that (3.22) hold at t'_i . Then by above lemma we have $(M, \omega(t'_i))$ converges in the Gromov-Hausdorff topology to the unique limit (M_∞, d_∞) . In particular we have

$$R(\omega(t'_i)) \leq C, \text{diam}(M, \omega(t'_i)) \leq C, C_S(M, \omega(t'_i)) \leq C$$

for some C independent of i , where R is the scalar curvature, C_S is the Sobolev constant.

Let $\tilde{\omega}_i(\tilde{t}) = (1 + \tilde{t})\omega(t'_i + \log(1 + \tilde{t}))$ be the solution to the Ricci flow

$$\frac{\partial}{\partial \tilde{t}} \tilde{\omega}_i = -\text{Ric}(\tilde{\omega}_i), \tilde{\omega}_i(0) = \omega(t'_i). \quad (3.25)$$

Under this rescalings, the metric $\omega(t_i) = (1 + \tilde{t}_i)^{-1} \tilde{\omega}_i(\tilde{t}_i)$ for some $\tilde{t}_i \in [e - 1, e^2 - 1]$ and $B_{r_i, \omega(t_i)}(x_i) = B_{\tilde{r}_i, \tilde{\omega}_i(\tilde{t}_i)}(x_i)$ for some $\tilde{r}_i \leq e$. Recall the μ functional of Perelman [20]

$$\mu(g, \tau) = \inf \int_M [\tau(R + |\nabla f|^2) + f - 2n] (4\pi\tau)^{-n} e^{-f} dv_g$$

for any Riemannian metric g and $\tau > 0$, where the infimum is taken over all $f \in C^\infty(M; \mathbb{R})$ with restriction $\int_M (4\pi\tau)^{-n} e^{-f} dv_g = 1$. The condition (3.24) implies that $\mu(\tilde{\omega}_i(\tilde{t}_i), \tilde{r}_i^2) \rightarrow$

$-\infty$ as $i \rightarrow \infty$; see [20, Section 4.1]. Then Perelman's monotonicity formula shows $\mu(\tilde{\omega}_i(0), \tilde{t}_i + \tilde{r}_i^2) \rightarrow -\infty$ as $i \rightarrow \infty$, where $\tilde{t}_i + \tilde{r}_i^2 \in [e - 1, 2e^2 - 1]$. But this can never happen because of the lower estimate of μ (cf. the estimate in [36]):

$$\mu(\tilde{\omega}_i(0), \tau) \geq \tau \inf R(\tilde{\omega}_i(0)) - \frac{n}{2} \ln \tau - n \ln C_S(\tilde{\omega}_i(0)) - C(n), \forall \tau \geq \frac{n}{8}. \quad (3.26)$$

So (3.24) can not hold, the lemma is proved. \square

The global Cheeger-Gromov convergence is a direct corollary of the following proposition.

Proposition 3.15. *For any sequence $t_j \rightarrow \infty$, (M, ω_{t_j}) converges along a subsequence in the Cheeger-Gromov sense to the limit (M_∞, d_∞) .*

Proof. By the regularity of manifolds with L^p bounded Ricci curvature, Theorem 2.3, and the uniform L^p estimate of Ricci curvature under the Kähler-Ricci flow, (3.2) and (3.3), we can find another sequence of times t'_j such that

$$t_j - \varepsilon_j \leq t'_j \leq t_j$$

where $\varepsilon_j \rightarrow 0$ as $j \rightarrow \infty$, and

$$(M, \omega_{t'_j}) \xrightarrow{d_{GH}} (M_\infty, d_\infty)$$

along a subsequence. On the other hand, by Gromov precompactness theorem [17] together with the local noncollapsing estimate (3.23), after passing to a subsequence, the manifolds $(M, \omega(t_j))$ also converge in the Gromov-Hausdorff topology to a compact limit

$$(M, \omega(t_j)) \xrightarrow{d_{GH}} (M'_\infty, d'_\infty).$$

It is remained to show that (M'_∞, d'_∞) is isometric to (M_∞, d_∞) . Actually we have

Claim 3.16. *There is a sequence of positive numbers $\delta_j \rightarrow 0$ as $j \rightarrow \infty$ such that the identity map defines a δ_j -Gromov-Hausdorff approximation from $(M, \omega(t'_j))$ to $(M, \omega(t_j))$.*

Proof of the Claim. Recall that by the smooth convergence of $\omega(t)$ on $M \setminus E$ and uniform volume noncollapsing (3.23) we have for any $\delta > 0$ one compact subset $K \subset M \setminus E$ and $\epsilon > 0$, $j_0 \gg 1$ such that

$$\epsilon \leq \inf_t \text{dist}_{\omega(t)}(K, E) \leq \sup_t \sup_{x \in E} \text{dist}_{\omega(t)}(x, K) \leq \delta, \quad (3.27)$$

and,

$$\text{vol}_{\omega(t)}(M \setminus K) \leq \delta, \forall t \geq t_{j_0}. \quad (3.28)$$

The later can be seen by simply the derivative estimate to volume

$$\frac{d}{dt} \int_{M \setminus K} \omega(t)^n = - \int_{M \setminus K} (\inf R + n) \omega(t)^n \leq C(\omega_0) e^{-t} \int_{M \setminus K} \omega(t)^n$$

where we used the estimate for the scalar curvature (3.21). For example, one may choose j_0 and K such that $C(\omega_0)e^{-j_0} \leq \ln 2$ and $\text{vol}_{\omega(t_{j_0})}(M \setminus K) \leq \frac{\delta}{2}$, then the volume estimate holds for any $t \geq t_{j_0}$. Moreover, by the convexity of the regular set $M \setminus E$ in M_∞ we may assume in addition that any minimal geodesic in $(M, \omega(t))$ with endpoints in K does not intersect the ϵ neighborhood of E . Also observe that there exists $C = C(\epsilon)$ independent of j such that

$$|\text{Ric}(\omega(t))|(x) \leq C, \forall \text{dist}_{\omega(t)}(x, E) \geq \frac{\epsilon}{2}. \quad (3.29)$$

Then, by the derivative estimate to distance function, cf. Lemma 8.3 in [20],

$$\frac{d}{dt} \text{dist}_{\omega(t)}(x, y) \geq -2(2n-1)(C\epsilon + \epsilon^{-1}), \forall x, y \in K.$$

Thus, we obtain the distance lower variation estimate

$$\text{dist}_{\omega(t_j)}(x, y) \geq \text{dist}_{\omega(t'_j)}(x, y) - 2(2n-1)(C\epsilon + \epsilon^{-1})\epsilon_j \geq \text{dist}_{\omega(t'_j)}(x, y) - \sqrt{\epsilon_j}$$

whenever j is large enough. On the other hand, let γ be any minimal geodesic in $(M, \omega(t'_j))$ connecting $x, y \in K$. By assumption $\text{dist}_{\omega(t)}(\gamma, E) \geq \epsilon$ for any t , so

$$\frac{d}{dt} \int_\gamma |\dot{\gamma}| ds \leq \int_\gamma |\text{Ric} + \omega| |\dot{\gamma}| ds \leq (C+n) \int_\gamma |\dot{\gamma}| ds$$

which gives the upper estimate to the distance distortion

$$\text{dist}_{\omega(t_j)}(x, y) \leq e^{(C+n)\epsilon_j} \text{dist}_{\omega(t'_j)}(x, y) \leq \text{dist}_{\omega(t'_j)}(x, y) + \sqrt{\epsilon_j}$$

whenever j is large enough. Finally, (3.27) shows that K is δ -dense in any $(M, \omega(t))$. Combining with the upper and lower distance variation estimate we get that the identity map defines an δ -Gromov-Hausdorff approximation between $(M, \omega(t_j))$ and $(M, \omega(t'_j))$ whenever j is large enough such that $\sqrt{\epsilon_j} \leq \delta$. The claim is proved. \square

Proposition 3.15 follows directly from the Claim. \square

We end the paper with a brief discussion about the algebraic structure of M_∞ . By Song's work [26], the limit space M_∞ coincides with the canonical model M_{can} , so it is a normal projective variety. On the other hand, using the Kähler-Ricci flow, we can produce a natural isomorphism from M_∞ to M_{can} through holomorphic sections of K_M^ℓ for some $\ell \gg 1$ such that K_M^ℓ is base point free, and consequently, give an alternative proof of the above result by J. Song. This can be done by choosing a family of orthonormal basis of $H^0(M, K_M^\ell)$ with respect to the Hermitian metric $h(t) = \omega(t)^{-n\ell}$, say $\{s_{i,t}\}_{i=0}^{N_\ell}$ where $N_\ell = \dim H^0(M; K_M^\ell) - 1$, which satisfies the ODE system

$$\frac{\partial}{\partial t} s_{i,t} = \sum_j b_{ij}(t) s_{j,t}. \quad (3.30)$$

In order to preserve the orthonormal property we choose

$$b_{ij} = \bar{b}_{ji} = -\frac{\ell-1}{2} \int_M (R+n) \langle s_{i,t}, s_{j,t} \rangle_{h(t)} \omega(t)^n. \quad (3.31)$$

Lemma 3.17. *There exists $C = C(\omega_0, \ell)$ such that*

$$|b_{ij}| \leq C e^{-t}.$$

Proof. First of all we notice that the section $s_{i,t}$ admits a uniform L^∞ bound. This can be seen by the uniform equivalence of the Hermitian metric $h(t) = e^{-\ell u(t)} \Omega^{-\ell}$, the volume form $\omega(t)^n = e^{u(t)} \Omega$ and the L^∞ estimate of holomorphic sections at time $t = 0$. Here we used the uniform C^0 estimate of u (3.8). Then we can estimate the integration as in the proof of Lemma 3.7, by $R + n \geq -C e^{-t}$ and $\int (R + n) \omega^n \leq C e^{-t}$,

$$|b_{ij}| \leq C \int |R + n| \omega^n \leq C \int (R + n + C e^{-t}) \omega^n + C e^{-t} \leq C e^{-t}$$

where $C = C(\omega_0, \ell)$. □

Suppose $s_{i,t} = \sum_j a_{ij}(t) s_{j,0}$ and denote by $A(t) = (a_{ij}(t))$ a Hermitian matrix, then $A(t) = e^{\int_0^t B(s) ds}$ where $B(t) = (b_{ij}(t))$. Thus the sections $\{s_{i,t}\}_{i=0}^{N_\ell}$ converge to a set of holomorphic sections $\{s_{i,\infty}\}_{i=0}^{N_\ell}$ which forms another basis of $H^0(M; K_M^\ell)$. It induces a morphism

$$\Phi : M \rightarrow M_{can} \subset \mathbb{C}P^{N_\ell}.$$

On the other hand, we also have a uniform gradient estimate to each $s = s_{i,0}$, cf. [26],

$$|\nabla^{h(t)} s|_{h(t) \otimes \omega(t)} = |\nabla^{\Omega^{-\ell}} s + \partial u \otimes s|_{h(t) \otimes \omega(t)} \leq C |\nabla^{\Omega^{-\ell}} s|_{\Omega^{-\ell} \otimes \chi} + C |\partial u|_{\omega(t)} |s|_{\Omega^{-\ell}} \leq C$$

where we used the uniform C^1 estimate of u and $\chi \leq C \omega(t)$, this leads to a convergence of $\{s_{i,t}\}_{i=0}^{N_\ell}$ under the Cheeger-Gromov convergence, so $\{s_{i,\infty}\}_{i=0}^{N_\ell}$ can also be seen as a set of holomorphic sections of $K_{M_\infty}^\ell$. It is obvious that $\{s_{i,\infty}\}_{i=0}^{N_\ell}$ is base point free on M_∞ , so it defines a map

$$\Phi_\infty : M_\infty \rightarrow \mathbb{C}P^{N_\ell}.$$

Finally, through construction of local peak sections, up to rising a power of ℓ , the map Φ_∞ separates points of M_∞ , so it defines a homeomorphism.

References

- [1] M. Anderson, *Ricci curvature bounds and Einstein metrics on compact manifolds*, J. Amer. Math. Soc. 2 (1989), 455-490.
- [2] M. Anderson, *Convergence and rigidity of manifolds under Ricci curvature bounds*, Invent. Math., 102 (1990), 429-445.
- [3] R. H. Bamler and Q. S. Zhang, *Heat kernel and curvature bounds in Ricci flows with bounded scalar curvature*, arXiv:1501.02191v2
- [4] S. Bando, A. Kasue and H. Nakajima, *On a construction of coordinates at infinity on manifolds with fast curvature decay and maximal volume growth*, Invent. Math. 97 (1989), no. 2, 313-349.

- [5] A. L. Besse, *Einstein manifolds*, Springer-Verlag Berlin Heidelberg GmbH
- [6] H.D. Cao, *Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds*, Invent. Math., 81 (1985), 359-372.
- [7] J. Cheeger, *Integral bounds on curvature, elliptic estimates and rectifiability of singular sets*, Geom. Funct. Anal., 13 (2003), 20-72.
- [8] J. Cheeger and T. H. Colding, *Lower bounds on the Ricci curvature and the almost rigidity of warped products*, Ann. Math., 144 (1996), 189-237.
- [9] J. Cheeger and T. H. Colding, *On the structure of spaces with Ricci curvature bounded below I*, J. Diff. Geom., 46 (1997), 406-480.
- [10] J. Cheeger and T. H. Colding, *On the structure of spaces with Ricci curvature bounded below II*, J. Diff. Geom., 54 (2000), 13-35.
- [11] J. Cheeger, T. H. Colding and G. Tian, *On the singularities of spaces with bounded Ricci curvature*, Geom. Funct. Anal., 12 (2002), 873-914.
- [12] T. H. Colding, *Ricci curvature and volume converge*, Ann. of Math., 145 (1997), 477-501.
- [13] T. H. Colding and A. Naber, *Sharp Hölder continuity of tangent cones for spaces with a lower Ricci curvature bound and applications*, Ann. of Math., 176 (2012), 1173-1229.
- [14] T. C. Collins and V. Tosatti, *Kähler currents and null loci*, arXiv:1304.5216v5
- [15] C. B. Croke, *Some isoperimetric inequalities and eigenvalue estimates*, Ann. Sci. Éc. Norm. Sup. Paris, 13 (1980), 419-435.
- [16] X.Z. Dai and G.F. Wei, *Comparison geometry for Ricci curvature*, preprint.
- [17] M. Gromov, *Metric structures for Riemannian and non-Riemannian spaces*. With appendices by M. Katz, P. Pansu and S. Semmes. Translated from the French by Sean Michael Bates. Birkhäuser Boston, Inc., Boston, MA, 2007.
- [18] B. Guo, *On the Kähler Ricci flow on projective manifolds of general type*, arXiv:1501.04239
- [19] B. Guo, J. song and B. Weinkovw, *Geometric convergence of the Kähler-Ricci flow on complex surfaces of general type*, arXiv:1505.00705
- [20] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arXiv:math.DG/0211159
- [21] P. Petersen and G.F. Wei, *Relative volume comparison with integral curvature bounds*, Geom. Funct. Anal., 7 (1997), 1031-1045.
- [22] P. Petersen and G.F. Wei, *Analysis and geometry on manifolds with integral Ricci curvature bounds, II*, Trans. AMS., 353 (2001), 457-478.

- [23] X.C. Rong and Y.G. Zhang, *Continuity of extremal transitions and flops for Calabi-Yau manifolds*, J. Diff. Geom., 89 (2011), 233-269.
- [24] S.C. Chen and M.C. Shaw, *Partial differential equations in several complex variables*, Amer. Math. Soc., Providence, RI; Intern. Press, Boston, MA, 2001.
- [25] M. Simon, *Some integral curvature estimates for the Ricci flow in four dimensions*, arXiv:1514.02623v1
- [26] J. Song, *Riemannian geometry of Kähler-Einstein currents*, arXiv:1404.0445
- [27] J. Song, *Riemannian geometry of Kähler-Einstein currents II: an analytic proof of Kawamata's base point free theorem*, arXiv:1409.8374
- [28] J. Song and G. Tian, *Bounding scalar curvature for global solutions of the Kähler-Ricci flow*, arXiv:1111.5681v1
- [29] G. Tian, *On Calabi's conjecture for complex surfaces with positive first Chern class*. Invent. Math., 101, (1990), 101-172.
- [30] G. Tian and Z.L. Zhang, *Regularity of Kähler-Ricci flows on Fano manifolds*, arXiv:1310.5897
- [31] G. Tian and Z. Zhang, *On the Kähler-Ricci flow on projective manifolds of general type*, Chinese Ann. Math. Ser. B, 27 (2006), 179-192.
- [32] H. Tsuji, *Existence and degeneration of Kähler-Einstein metrics on minimal algebraic varieties of general type*, Math. Ann., 281 (1988), 123-133.
- [33] Q.S. Zhang, *A uniform Sobolev inequality under Ricci flow*, Inter. math. Res. Notices, 2007.
- [34] Y.G. Zhang, *Miyaoka-Yau inequality for minimal projective manifolds of general type*, Proc. Amer. Math. Soc. 137 (2009), no. 8, 2749C2754.
- [35] Z. Zhang, *Scalar curvature bound for Kähler-Ricci flows over minimal manifolds of general type*, Int. Math. Res. Not., 2009, 3901-3912.
- [36] Z.L. Zhang, *Compact blow-up limits of finite time singularities of Ricci flow are shrinking Ricci solitons*, C. R. Acad. Sci. Paris, Ser. I 345 (2007), 503-506.